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LETTER TO THE EDITOR

A new Holstein–Primakoff realisation of sp(4, ℝ)

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Abstract. A new Holstein-Primakoff type realisation of the non-compact Lie algebra sp(4, ℝ) is given in a completely explicit, analytic and closed form. The construction is related to the representations where the Casimir invariants of the subalgebras $\mathrm{sp}_{1,2}(2, \mathbb{R})$ are diagonal, rather than the invariants of the maximal compact subalgebra u(2).

After the work of Mlodinow and Papanicolaou (1980, 1981) the importance of Holstein-Primakoff (1940) type realisations of the irreducible representations of the non-compact symplectic Lie algebras sp(2d, ℝ) is generally accepted, especially in applications to many-particle quantum problems. Extensive work has been done to obtain such realisations (Moshinsky 1985, Deenen and Quesne 1985). The common approach was to look for basis of state vectors associated with the irreps in the chain of groups $\mathrm{Sp}(2d, \mathbb{R}) \supset \mathrm{U}(d)$. However, the known results are not given in a completely analytic form. Numerical computation must be performed at some stage. In the present letter I would like to draw attention to another choice of the basis, namely those associated with the chain $\mathrm{Sp}(2d, \mathbb{R}) \supset [\mathrm{Sp}(2, \mathbb{R})]^{\otimes d}$. In particular, a completely analytic Holstein-Primakoff realisation for the case $d = 2$ will be given. It will be related to the representations where the Casimir invariants $J^2_{1,2}$ of the subalgebras $\mathrm{sp}_{1,2}(2, \mathbb{R})$ are diagonal. Then, the change to another basis could be obtained, if necessary, by means of a unitary transformation.

As is well known (Castaños *et al* 1985), a set of creation η_{ir} and annihilation ξ_{ir} operators ($i = 1, 2$; $r = 1, 2, \ldots, n$) of a system of $2n$ Bose oscillators can be used to give the sp(4, ℝ) generators in the form

$$C_{ij} = \sum_{r=1}^{n} \eta_{ir}\xi_{jr} + \tfrac{1}{2}n\delta_{ij} \tag{1a}$$

$$B_{ij} = \sum_{r=1}^{n} \xi_{ir}\xi_{jr} \qquad B^+_{ij} = \sum_{r=1}^{n} \eta_{ir}\eta_{jr}. \tag{1b}$$

The irreps of sp(4, ℝ) can be parametrised by the pairs $\lambda_1 + \frac{1}{2}n, \lambda_2 + \frac{1}{2}n]$, where the integers λ_1, λ_2 satisfy the inequalities $0 \leqslant \lambda_1 \leqslant \lambda_2$. The quantities $\lambda_i + \frac{1}{2}n$ are the eigenvalues of the weight operators C_{ii} ($i = 1, 2$) in the lowest weight state vector $|\mathrm{LW}\rangle$, which is defined by

$$C_{21}|\mathrm{LW}\rangle = 0 \tag{2a}$$

$$B_{ij}|\mathrm{LW}\rangle = 0. \tag{2b}$$

All discrete series of irreducible representations of sp(4, ℝ) in a separable Hilbert space were found and classified by Evans (1967). It was done using basis of state vectors parametrised by the eigenvalues of $J^2_{1,2}$ and C_{11}, C_{22}. Each irrep was identified by the pair (q, s), where $q=(\lambda_2+\lambda_1+n)/2$ and $s=(\lambda_2-\lambda_1)/2$. For all of them, it can be written

$$J_i^2|q, s:j_1, m_1, j_2, m_2\rangle = j_i(j_i-1)|q, s:j_1, m_1, j_2, m_2\rangle \quad (3)$$

$$C_{ii}|q, s:j_1, m_1, j_2, m_2\rangle = 2m_i|q, s:j_1, m_1, j_2, m_2\rangle \quad (4)$$

where

$$J_i^2 \equiv [C_{ii}^2 - \tfrac{1}{2}(B_{ii}^+B_{ii}+B_{ii}B_{ii}^+)]/4 \qquad i=1,2$$

and $\{|q, s:j_1, m_1, j_2, m_2\rangle\}$ is a complete basis of orthonormalised state vectors.

There are four different series in the classification given by Evans. But it will be enough to limit the discussion to one of them, which seems to be the most useful in physical applications. The other series could be treated in a similar fashion. Thereafter, I will deal only with the series of irreps given by the values

$$s=0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

$$q>s+\tfrac{1}{2} \qquad \text{if } s=0, \tfrac{1}{2}$$

$$q>s+1 \qquad \text{if } s=1, \tfrac{3}{2}, \ldots.$$

In this case, the state vectors are parametrised within a given irrep by the values $j_1=(q+l+\delta)/2$, $m_1=j_1+\nu_1$, $j_2=(q+l-\delta)/2$, $m_2=j_2+\nu_2$, where $\nu_1, \nu_2, l=0, 1, 2, \ldots$, and $\delta=s, s-1, \ldots, -s$. The present notation is that of Inaba *et al* (1982) where, excluding some 'singular' cases, the matrix elements of the ten generators of sp(4, $\mathbb{R}$) are listed. The 'singular' cases arise when $q=s+1, s+2$. To be brief I exclude these cases too and assume $q\neq s+1, s+2$.

The range of parameters ν_1, ν_2, l and δ shows that a Holstein–Primakoff realisation of sp(4, $\mathbb{R}$) can be looked for in the representation space of a Lie algebra w(3)$\oplus$su(2), where w(3) is the Weyl algebra of three bosons oscillators. Then, let us start with the assumption of the one-to-one correspondence (for given values of q and s)

$$|q, s:j_1=\tfrac{1}{2}(q+l+\delta), m_1=j_1+\nu_1, j_2=\tfrac{1}{2}(q+l-\delta), m_2=j_2+\nu_2\rangle \leftrightarrow |\nu_1, \nu_2, l\rangle \otimes |s, \delta\rangle \quad (5)$$

where $\{|\nu_1, \nu_2, l\rangle\}$ is an orthogonal basis for w(3), parametrised by the eigenvalues of the occupation number operators of three independent boson oscillators $N_a\equiv b_a^+b_a$, $a=1, 2, 3$, and $\{|s, \delta\rangle\}$ is a standard basis of su(2), whose generators will be denoted by S_1, S_2, S_3. The lowest state vector corresponds to the values $\nu_1=\nu_2=l=0$, $\delta=-s$.

The analysis of the matrix elements of the sp(4, $\mathbb{R}$) generators, through the assumed one-to-one correspondence, allows us to state that the following operators give the Holstein–Primakoff realisation for the mentioned series of irreps ($q=s+1, s+2$ excluded):

$$C_{11}=q+2N_1+N_3+S_3 \quad (6)$$

$$C_{22}=q+2N_2+N_3-S_3 \quad (7)$$

$$C_{12}=b_3^+b_2(q+N_1+N_3+S_3)^{1/2}F+b_1^+b_3(q+N_2+N_3-S_3-1)^{1/2}F'$$
$$+S_+(q+N_1+N_3+S_3)^{1/2}(q+N_2+N_3-S_3-1)^{1/2}G+b_1^+b_2S_-G' \quad (8)$$

$$B_{11}^+=2b_1^+(q+N_1+N_3+S_3)^{1/2} \quad (9)$$

$$B_{22}^+=2b_2^+(q+N_2+N_3-S_3)^{1/2} \quad (10)$$

$$B^+_{12} = b^+_3(q+N_1+N_3+S_3)^{1/2}(q+N_2+N_3-S_3)^{1/2}F + b^+_1 b^+_2 b_3 F'$$
$$+ b^+_2 S_+(q+N_1+N_3+S_3)^{1/2}G + b^+_1 S_-(q+N_2+N_3-S_3)^{1/2}G' \quad (11)$$

where $S_\pm = S_1 \pm \mathrm{i}S_2$, $F \equiv F(N_3, S_3)$, $F' \equiv F(N_3-1, S_3)$, $G \equiv G(N_3, S_3)$ and $G' \equiv G(N_3, S_3-1)$, with

$$F(N_3, S_3) = \left(\frac{(q+s+N_3)(q-s+N_3-1)(2q+N_3-2)}{(q+N_3+S_3)(q+N_3+S_3-1)(q+N_3-S_3)(q+N_3-S_3-1)}\right)^{1/2} \quad (12)$$

$$G(N_3, S_3) = \left(\frac{(q-S_3-2)(q+S_3-1)}{(q+N_3+S_3)(q+N_3+S_3-1)(q+N_3-S_3-1)(q+N_3-S_3-2)}\right)^{1/2}. \quad (13)$$

The remaining operators follow by Hermitian conjugation.

Now, it can be directly verified that the operators (6)–(13) satisfy the $\mathrm{sp}(4,\mathbb{R})$ commutation relations. This realisation has the explicit, analytic and closed form which becomes specially useful in physical applications. A non-trivial example is the so-called $1/N$ expansion of rotational invariant two-particle Hamiltonians. This was developed by Mlodinow and Papanicolaou (1981) for the case of s-wave states (total angular momentum $L=0$). Now, the extension to arbitrary values L of total angular momentum can be given in the basis of appropriate selection of the irreps of $\mathrm{sp}(4,\mathbb{R})$ associated with given values of L. As was stated by Mlodinow and Papanicolaou (1981), there is a definite relation between the total angular momentum and the Casimir invariants of $\mathrm{sp}(4,\mathbb{R})$. With the choice $n=3$ in (1*a*) and (1*b*) it can be written

$$L^2 = 2s(s+1) + 2q(q-3) + 9/2 \quad (14)$$

where $L^2 = L_1^2 + L_2^2 + L_3^2$ and

$$L_r = -\mathrm{i} \sum_{r',r''=1}^{3} \varepsilon_{rr'r''}(\eta_{1r'}\xi_{1r''} + \eta_{2r'}\xi_{2r''}). \quad (15)$$

A systematic choice of irreps with given eigenvalues $L(L+1)$ of L^2 can be obtained making $s = L/2$ and $q = s + \frac{3}{2} = (L+3)/2$. The '$1/N$' expansion can be obtained expressing the Hamiltonian as an explicit function of the $\mathrm{sp}(4,\mathbb{R})$ generators and then expanding it in powers of a small parameter $x = q^{-1/2} = [2/(L+3)]^{1/2}$. It is clear that the above series of irreps is enough for this purpose. Further discussion of such expansions could exceed the scope of the present letter and will be given elsewhere.

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